ON THE STRUCTURE OF SHOCK WAVES IN MAGNETOHYDRODYNAMICS WITH ARBITRARY DISSIPATION LAW

(O STRUKTURE UDARNYKH VOLN V MAGNITNOI GIDRODINAMIKE PRI PROIZVOL'NOM ZAKONE DISSIPATSII)

PMM Vol.26, No.2, 1962, pp. 273-279

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(Received December 15, 1961)

One-dimensional steady flow is considered for a viscous, heat-conducting gas with finite electrical conductivity.

Under certain hypotheses (1) on the equation of state, it is shown that there exist flows which represent evolutionary [1] fast and slow shock waves of not too large amplitudes. The fast shocks given by such flows turn out to be unique.

For the description of the dissipative process, the principle of Onsager is used, in which the rate of entropy increase d_iS/dt is considered to be positive, if at least one of the space derivatives of the flow parameters is different from zero. The method of the proof is based on the results of [2, 3], which show examples of systems of partial differential equations in two and three unknowns and study solutions of the travelling wave type. For more special dissipation laws, the problem of the structure of oblique magnetohydrodynamic shock waves has already been considered in [4-9].

We shall assume that the equation of state of the gas

$$p = p(V, S) \qquad (V = 1/\rho, \text{ the specific volume})$$

satisfies the conditions

$$p_V' < 0, \quad p_{VV}'' > 0, \quad p_S' > 0 \tag{1}$$

In considering one-dimensional (along x) steady gas flows with an electric field, we may use the conservation laws, expressing the viscous stresses τ_{xx}, τ_{xy}, τ_{xz} and the heat flow Q in the medium in terms of the parameters characterizing the gas flow and electric field:

$$V, \quad T, \quad u, \quad v, \quad w, \quad H_y, \quad H_z, \quad H_x = \text{const}, \quad E_y = \text{const}, \quad E_z = \text{const}$$

This permits the calculation of the entropy flow

$$P = \frac{Q}{T} + mS = \frac{m}{T}\left[\frac{H_y^2 V}{8\pi} + \frac{H_z^2 V}{8\pi} + \frac{m^2V^2}{2} + \frac{v^2}{2} + \frac{w}{2} - f(V, T) - \right.$$
$$\left. - H_0 H_y v - H_0 H_z w - JV + \varepsilon\right] \tag{2}$$

Here m is the mass flow; $f(V, T)$ the Helmholtz free energy per unit mass; J the flow of the x-component of the momentum; ε the energy flow divided by m

$$df = -SdT - pdV, \qquad H_0 = \frac{H_x}{4\pi m}, \qquad E_0 = \frac{cE_z}{4\pi m} \text{ (} c \text{ is the light speed)}$$

The coordinate system is so chosen that E_y as well as the y- and z-components of the momentum equal zero.

Obviously, the flux of entropy flow is connected with its rate of increase thus

$$\frac{dP}{dx} = \frac{d_i S}{dt} \tag{3}$$

According to the principle of Onsager

$$\frac{d_i S}{dt} = \sum_i J_i X_i \tag{4}$$

where J_i are the generalized fluxes, and X_i the generalized forces. Taking as X_i the derivatives $\dot{q}_i = dq_i/dx$ of the quantities V, v, w, T, H_y, H_z, which we denote by q_i, and using the identity

$$\frac{dP}{dx} = \sum_i \frac{\partial P}{\partial q_i} \dot{q}_i$$

we obtain from Equations (3) and (4)

$$\partial P / \partial q_i = J_i \tag{5}$$

Let us assume, as is usually done in thermodynamics of irreversible processes, that the J_i are linear functions of X_i

$$J_i = \sum_j L_{ij} X_j = \sum_i L_{ij} q_i$$

such that the quadratic form

$$D = \sum_{ij} L_{ij}\dot{q}_i\dot{q}_j$$

for arbitrary $\dot{q}_k$ satisfies the inequality $D \geqslant 0$. In what follows, the coefficients L_{ij} will be assumed to be continuously differentiable functions of q_k.

Moreover, we shall assume that the matrix L_{ij} is nonsingular, i.e. that $D > 0$ if any one of the $\dot{q}_i \neq 0$. Substituting the expression for J_i into Equation (5), we obtain a system of equations satisfied by the functions $q_i(x)$ in one-dimensional steady flow

$$\sum_j L_{ij}\,\dot{q}_j = \frac{\partial P}{\partial q_i} \tag{6}$$

If the matrix L_{ij} is symmetric, which is generally not the case if a magnetic field is present, then Equation (6) may be written in the form

$$\frac{1}{2}\frac{\partial D}{\partial \dot{q}_i} = \frac{\partial P}{\partial q_i} \tag{7}$$

The equations describing the steady magnetohydrodynamic flows may be immediately reduced to the form (6) or (7). In the case $H_z = 0$, $w = 0$, and the matrix L_{ij} is diagonal, Equations (7) were obtained in [5], in which, under the indicated restrictions, the existence and uniqueness of the solution, representing the structure of a fast shock wave was proved.

A uniform translational flow ($\dot{q}_i = 0$) corresponds to a singular point A_α of the system (6), or equivalently, to a stationary point of the function $P(q_i)$. Thus the solution to the problem of shock-wave structure must be represented by an integral curve of the system (6) in the q_i space, connecting the singular points of this system.

One readily convinces oneself that all the singular points of system (6) lie in the plane $H_z = 0$, $w = 0$, if

$$E_0 \neq 0, \quad \text{or} \quad H_\tau\left(u - \frac{H_x}{\sqrt{4\pi\rho}}\right)\bigg|_{x=\pm\infty} \neq 0 \qquad (\mathbf{H}_\tau = H_y\mathbf{e}_y + H_z\mathbf{e}_z)$$

The case

$$H_\tau\left(u - \frac{H_x}{\sqrt{4\pi\rho}}\right) = 0 \quad \text{for } x = \pm\infty$$

correspond either to gasdynamical shock waves, or to shock waves lying on the boundary of evolutionarity. Therefore, we shall assume throughout that $E_0 \neq 0$. For this, as follows from [5], $P(q_i)$ possesses not more than four stationary points A_1, A_2, A_3, A_4, in which

$$P(A_1) < P(A_2) < P(A_3) < P(A_4)$$
$$a_+(A_1) < u(A_1), \qquad a_A(A_2) < u(A_2) < a_+(A_2) \tag{8}$$
$$a_-(A_3) < u(A_3) < a_A(A_3) \qquad u(A_4) < a_-(A_4)$$

Here a_+, a_A and a_- are the propagation speeds of the fast, Alfven, and slow small-disturbance waves, and $u = mV$. In addition, to fast waves correspond transitions $A_1 \to A_2$, and to slow waves correspond $A_3 \to A_4$. We observe, that the points A_1 and A_2 lie in the region $V > 4\pi H_0^2$, while the points A_3 and A_4 in the region $V < 4\pi H_0^2$, since $u^2 4\pi H_0^2 = a_A^2 V$.

We consider the behavior of the function $P(q_i) - P(A_\alpha)$ in the neighborhood of the singular point A_α. If we retain in this difference only the quadratic terms in the differences in coordinates (the linear terms vanish since A_α is a stationary point of the function $P(q_i)$), we may show that the quadratic form so obtained is non-singular and that its trace equals $8 - 2\alpha$. This is easily seen by reducing the quadratic form representing the dominant part of the difference $P(q_i) - P(A_\alpha)$ to a sum of squares (as was done in [5] for the case $H_z = 0$, $w = 0$).

The behavior of the integral curves of the system (6) in the neighborhood of the singular points A_α is determined by the linearized system of equations, which are obtained by substituting the dominant part of $P(q_i) - P(A_\alpha)$ into the right-hand side of (6). Since the system under consideration is dissipative in the sense of [10], then from the results of this work and from the inequalities (8), it follows* that out of the six eigenvalues of the linearized system, $7 - \alpha$ have positive real parts, and $\alpha - 1$ have negative real parts. Thus, at each singular point A_α, there is a $7 - \alpha$ dimensional surface consisting of all the integral curves issuing from the point, and an $\alpha - 1$ dimensional surface consisting of all the integral curves entering the point.

Consider the surface $P(q_i) = C$, $C = \text{const}$. The portion of the surface lying in the region $V > 0$, $T > 0$, contains the point at infinity. Actually, the intersection of the surface $P(q_i) = C$, which is represented

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* All conclusions of [10] remain true in the case when in some of the linear first order equations comprising the system, the derivatives of the unknown functions with respect to time are missing. In the case under consideration the ideal system of the highest rank does not contain the wave speed U; therefore, the number of roots with positive real parts changes only when U, while varying, crosses values of speeds of propagation of weak magnetohydrodynamic discontinuities.

by the equation

$$2P(q_i)=\frac{m}{T}\Big[\frac{V}{4\pi}\Big(H_y-\frac{4\pi H_0}{V}v+\frac{4\pi E_0}{V}\Big)^2+\frac{V}{4\pi}\Big(H_z-\frac{4\pi H_0}{V}w\Big)^2+$$

$$+\Big(1-\frac{4\pi H_0^2}{V}\Big)\Big(v+\frac{4\pi H_0E_0}{V-4\pi H_0^2}\Big)^2+\Big(1-\frac{4\pi H_0^2}{V}\Big)w^2\Big]+2F(V,T)=2C \quad (9)$$

$$F(V,T)=\frac{m}{T}\Big[\varepsilon+\frac{m^2V^2}{2}-\frac{2\pi E_0^2}{V-4\pi H_0^2}-JV-f(V,T)\Big]$$

with the plane V = const, T = const defines a second order surface in a four-dimensional space, which for $V < 4\pi H_0^2$ extends to infinity.

Let us assign a large negative value to C and then let it increase. The surface $P(q_i) = C$ will then change, but its topological character will only change when C passes through the values $C = P(A_\alpha)$. By virtue of the inequalities (8), the first change in the topological character of the surface $P(q_i) = C$ will occur when, in the process of C increasing, the surface $P(q_i) = C$ passes through the point A_1 (if it exists for given values of E_0, H_0, J, ε). Since the point A_1 is a node for the system (6), then for $C = P(A_1) + \delta$, δ being a sufficiently small positive number, the surface $P(q_i) = C$ in the neighborhood of the point A_1 will be a closed surface containing the point A_1 in its interior. The surface $P(q_i) = C$ contains the point at infinity, therefore, it must have another branch extending to infinity. The region $P(q_i) \leqslant C$ expands as C increases. The closed branch of the surface $P(q_i) = C$, containing the point A_1 inside, does not go out of the region $V > 4\pi H_0^2$, $T > 0$. Actually, the intersection of the closed branch of $P(q_i) = C$ with any plane is a closed surface. However, for $V < 4\pi H_0^2$, the intersection of the surface $P(q_i) = C$ with the plane V = const, T = const does not contain a closed branch. Moreover, the intersection of the surface $P(q_i) = C$ with the plane $T = 0$ does not depend on the value of C. For values of C close to $P(A_1)$, the closed branch of the surface $P(q_i) = C$ containing the point A_1 inside does not intersect the plane $T = 0$; consequently, this branch does not intersect this plane for any C.

Since grad $P(q_i)$ nowhere tends to infinity, then for increasing C, both branches of the surface move with non-zero velocity toward each other and must be joined at some value C. This connecting must occur at a stationary point of the function $P(q_i)$, which is the point A_2, since out of the remaining singular points it is the only one lying in the region $V > 4\pi H_0^2$, and it is the only one that can describe the joining of the two parts of the surface $P(q_i) = C$. We assume that the shock waves are not too strong, and therefore the values $P(A_1)$ and $P(A_2)$ are sufficiently close to each other, so that as C varies in the interval $P(A_1) < C < P(A_2)$, the closed branch of the surface $P(q_i) = C$ does not extend to infinity.

Since the singular point A_1 is a node of the system (6), the integral curves fill the entire space in the neighborhood of the point A_1, and every point on the closed branch of the surface $P(q_i) = C$ with $P(A_1) < C < P(A_2)$ may be joined to the point A_1 by an integral curve. In addition, one integral curve issuing from the point A_1 reaches the point A_2, when at $C = P(A_2)$ some point on the closed branch of the surface $P(q_i) = C$ arrives at the point A_2. This proves the existence and uniqueness of the solution, representing the structure of a fast shock wave.

Let us now consider the question of the existence of a solution, representing the structure of a slow shock wave. If for given values of the parameters E_0, H_0, J, ε, slow shock waves may be realized, then as C varies, the surface $P(q_i) = C$ must encounter a singular point A_3.

We note that the function $F(V, T)$ has a minimum at the point A_3, and a saddle point at the point A_4. Actually, we have the equality

$$P(q_i) - P(A_\alpha) = \frac{2m}{T}\left[\frac{V}{4\pi}\left(H_y - \frac{4\pi H_0}{V}v + \frac{4\pi E_0}{V}\right)^2 + \frac{V}{4\pi}\left(H_z - \frac{4\pi H_0}{V}w\right)^2 + \right.$$
$$\left. + \left(1 - \frac{4\pi H_0^2}{V}\right)\left(v + \frac{4\pi H_0 E_0}{V - 4\pi H_0^2}\right)^2 + \left(1 - \frac{4\pi H_0^2}{V}\right)w^2\right] + F(V, T) - F(A_3)$$

since at the stationary points, each of the squares in the square brackets equals zero. Expanding in the neighborhood of the point A_3 the function $P(q_i) - P(A_\alpha)$ in powers of the differences $V - V(A_3)$ and $T - T(A_3)$, confining our attention to the quadratic terms, and transforming the obtained quadratic form into a sum of squares, we get a representation of the dominant part of the difference $P(q_i) - P(A_3)$ in the neighborhood of the point A_3 as a sum of squares of the linear combinations of the differences $q_i - q_i(A_3)$. According to the previous statements, two of the coefficients in this expansion must be negative, and the rest positive. Since there are contained in the brackets two negative terms ($V(A_3) < 4\pi H_0^2$), then $F(V, T) - F(A_3)$ in the neighborhood of A_3 is represented as the sum of two positive terms, i.e. $F(V, T)$ has a minimum at the point A_3. Similarly, we show that the function $F(V, T)$ has a saddle point at the point A_4. There are no other stationary points of the function $F(V, T)$ in the region $0 < V < 4\pi H_0^2$, because otherwise the function $P(q_i)$ will also have other stationary points in this region, in addition to A_3 and A_4.

Since the function $F(V, T)$ has a minimum at the point A_3, the curve $F(V, T) = F(A_3) + \delta$ has a closed branch in the neighborhood of the point A_3, inside which $F(V, T) < F(A_3) + \delta$, and outside which $F(V, T) > F(A_3) + \delta$. At $\delta = 0$, this branch is represented by the single point A_3. The region $F(V, T) \leqslant C$ grows as C increases; moreover, from the fact that inside the region $4\pi H_0^2 > V > 0$, $T > 0$, the vector grad $F(q_i)$ nowhere tends to infinity, it follows that as C changes, the curve $F(V, T) = C$ moves with nonzero velocity. In this connection, the closed branch of the curve will remain closed, so long as it does not encounter a stationary point and does not go to infinity.

We shall assume that the intensity of the shock waves are not too great, so that the points A_3 and A_4 are not too far apart, and the closed branch will encounter the point A_4 during its motion. Since the point A_4 is a saddle point, then as C passes through the value $C = P(A_4)$, the two branches of the curve $F(V, T) = C$ will join at the point A_4.

We shall show that for $C = P(A_3)$, it is possible to construct in the region $P(q_i) < C$ of the q_i-space a two-dimensional closed surface Σ, which cannot contract to a point by continuous deformation in this region. To this end, we set $C = P(A_3)$ in Equation (9), and we consider the surface, described by an ellipse at the throat of the four-dimensional hyperboloid $P(q_i) = C$, $V = \text{const}$, $T = \text{const}$ for values of V and T varying along some curve a, connecting in the V-T plane the point A_3 with some other branch of the curve $F(V, T) = C$. At the end points of the curve a, the diameter of the ellipse becomes zero, and at the intermediate points it assumes positive values. Therefore, the ellipse sweeps out some two-dimensional closed surface, which, obviously, cannot shrink to a point by continuous deformation in the region $P(q_i) < C$. For $P(A_3) < C < P(A_4)$, each curve in the V-T plane, joining the two branches of the curve $F(V, T) = C$, corresponds to some two-dimensional surface in the six-dimensional space which cannot shrink to a point in the region $P(q_i) \leqslant C$.

We observe that as $C \to \infty$ all the finite points in the region $V > 0$, $T > 0$, of the q_i-space will satisfy $P(q_i) \leqslant C$. Consequently, the surface constructed above may be continuously contracted into a point for sufficiently large values of C in the region $P(q_i) \leqslant C$. From this, it follows that the topological type of the region $P(q_i) \leqslant C$ changes as C increases, and that the surface $P(q_i) = C$ must pass through a stationary point of the function $P(q_i)$ as C increases. This point is the point A_4, since among all the available stationary points A_α of the function $P(q_i)$, only at the point A_4 does the inequality $P(A_\alpha) > P(A_3)$ obtain, and only the point A_4 has the property that some two-dimensional surface, that cannot be shrunk to a point in the region $P(q_i) \leqslant C$ for $C < P(A_\alpha)$, can be shrunk to a point continuously when $C > P(A_\alpha)$.

Now it is easy to convince ourselves that at least one integral curve of the system (6) connects the singular points A_3 and A_4. In fact, let us consider the intersection of the three-dimensional surface consisting of the integral curves entering the singular point A_4, with the surface

$P(q_i) = C$ with $C = P(A_4) - \delta$, δ being a sufficiently small positive number. This intersection will turn out to be a two-dimensional surface $\Sigma^*(C)$, which cannot be continuously contracted to a point in the region $P(q_i) \leqslant C$ when $C < P(A_4)$, and which contracts to the point A_4 when $C \to P(A_4)$. From this, it follows that the surface $\Sigma^*(C)$ is homologous mod 2 to the surface Σ in the region $P(q_i) < C$ (i.e. in this region we can construct a three-dimensional surface bounded by $\Sigma^*(C)$ and Σ). This follows from the fact that passing through a simple stationary point, the number of homologically independent (mod 2) cycles in the region $P(q_i) \leqslant C$ changes from unity [11]. It is readily seen that the surfaces $\Sigma^*(C)$ and Σ may be continuously deformed into each other in the region $P(q_i) \leqslant C$.

As C varies, the surface $\Sigma^*(C)$ will deform continuously. This follows from the continuity and differentiability of the dissipative coefficients L_{ik} and from the positive-definiteness of the quadratic form D, insuring finite, nonzero angles between the surface $P(q_i) = C$ and the integral curve.

We shall consider the intensity of the shock waves to be not too large, so that as C varies from $P(A_4)$ to $P(A_3)$, the surface $\Sigma^*(C)$ neither leaves the region $V > 0$, $T > 0$, nor reaches infinity. In this case, the surface $\Sigma^*(C)$ for arbitrary values of C in the interval $P(A_3) < C < P(A_4)$ remains a closed surface and may be obtained from the surface Σ by continuous deformation in the region $P(q_i) \leqslant C$. As follows from the form of the surface $P(q_i) = P(A_3) + \delta$ in the neighborhood of the point A_3, any two-dimensional surface lying in the region $P(q_i) \leqslant P(A_3) + \delta$ and obtainable from the surface Σ (which passes through the point A_3) by continuous deformation in the region $P(q_i) \leqslant P(A_3) + \delta$, cannot be farther from A_3 than by a distance of the order $\sqrt{\delta}$. Consequently, for $C = P(A_3)$, the surface $\Sigma^*(C)$ passes through the point A_3. This proves that there exists at least one integral curve connecting the points A_3 and A_4.

In conclusion, the author thanks S.K. Godunov and G.A. Liubimov for discussions contained in the work and G.Ia. Liubarskii for kindly offering the author the possibility to become acquainted with his paper before its publication.

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Translated by C.K.C.